D-12/2110 5571/NJ

BMH-301/MC-301 Analysis-I

Semester-III

Note: Attempt any four questions. All questions carry equal marks.

Q1. i) Let $x \geq 0$, then prove that for every integer $n \geq 1$ there is a finite decimal $r_n = a_0 a_1 \dots a_n$ such that $r_n \leq x < r_n + \frac{1}{10^n}$. (8)

ii) State and prove Minkowski's inequality. (9.5)

Q2. i) Let $f: S \to T$ be a function and let $X \subseteq S$, $Y_1, Y_2 \subseteq T$, then prove that

a) $X \subseteq f^{-1}(f(x))$ b) $f(f^{-1}(Y_1)) \subseteq Y_2$ c) $f^{-1}(Y_1 \cup Y_2) = f^{-1}(Y_1) \cup f^{-1}(Y_2)$. (9)

ii) Define equinumerous set. Prove that every subset of a countable set is countable. (8.5)

Q 3. i) Prove that arbitrary union of open sets is open. (8)

ii) Let $A \subset \mathbb{R}^n$ and let F be an open covering of A. Prove that there exist a countable sub collection of F which also covers A. (9.5)

Q4. i) Prove that every non empty set S in $\mathbb{R}^1$ is the union of a countable collection of disjoint component intervals of S. (8)

ii) Prove that for any subset S of M the following statements are equivalent: (9.5)

a) S is closed in M.

b) S contains all its adherent points.

c) S contains all its accumulation points.

d) $S = \bar{S}$.

Q5. i) State and prove necessary and sufficient condition (in terms of open sets) for a function to be continuous. (8)

ii) Let $f: S \to \mathbb{R}^K$ be a function from a metric space S to $\mathbb{R}^K$. Prove that if f is continuous on a compact subset X of S then f is bounded on X. (9.5)

Q6. i) Define a Cauchy sequence . Prove that in Euclidean space $\mathbb{R}^K$ every Cauchy sequence is convergent. (9.5)

ii) Let $f: S \to T$ be a continuous function from one metric space (S,d_s) to another (T,d_T). Let A be a compact subset of S and assume that f is continuous on A. Prove that f is uniformly continuous on A. (8)

Q7. i) Prove that every open connected set in $\mathbb{R}^n$ is arcwise connected. (8)

ii) Let f be strictly increasing function on a set S in $\mathbb{R}$. Then prove that f^{-1} exist and is strictly increasing on f(S). (9.5)

Q8.i) If a bounded set S in $\mathbb{R}^n$ contains infinitely many points, then prove that there is atleast one point in $\mathbb{R}^n$ which is an accumulation point of S. (12.5)

ii) If x is an accumulation point of S, then prove that every n-ball B(X) contains infinitely many points of S. (5)

Q9. i) For arbitrary real x and y , prove that $|x+y| \leq |x|+|y|$. (17.5)

ii) Prove that $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$.

iii) Prove that every uniformly continuous function is continuous but the converse is not true

iv) Determine whether the $\lim_{(x,y)\to(0,0)} f(x,y) = \frac{x^2-y^2}{x^2+y^2}$, exist or not.

v) Define jump discontinuity and removable discontinuity. Give an example of a function which has removable jump discontinuity at 0.

vi) Determine all the accumulation points of the set of all rational numbers and decide whether the set is open or closed.