7411/N

DIFFERENTIABLE MANIFOLDS
MM-601/ AMC-308

Time allowed: 2 hours

Max. Marks: 70

Note: Attempt four questions in all carrying 17.5 marks each .

Q.1. Relate the concept of a tangent vector to a vector field. Prove that the set of all vector fields on a manifold M forms a vector space over $C^{\infty}(M)$.

Q.2. Discuss the geometry of a manifold with maximal atlas in terms of local coordinates. Elaborate the same with help of an example.

Q.3. Using the theory of submanifolds, prove that the induced connection of a Riemannian manifold is Riemannian.

Q.4. Discuss the theory of the basis of tensor product and wedge product. Hence show that for an n dimensional vector space V, if $r > n$, then $\wedge^r(V^*)$ is the zero space.

Q.5. Define the Lie-derivative of a one form and a two form. Hence show that $L_X = C_X \circ d + d \circ C_X$, where C_X denotes the contraction map, d the exterior derivative and L_X the Lie-derivative with respect to X.

Q.6. State and prove the fundamental theorem of Riemannian geometry.

Q.7. Discuss the geometry of the linear function instrumental in defining the sectional curvature of a plane section of the tangent space at a point of the manifold. Hence prove Schur's theorem.

Q.8. State and prove Bianchi's second identity for the Riemannian curvature tensor.

Q.9. (a) Explain the idea of an integral curve of a vector field as a solution to an initial value problem.Hence determine the integral curve of a vector field in $\Re^2$ given by $x^2\dfrac{\partial}{\partial x^1} - (x^2)^3\dfrac{\partial}{\partial x^2}$.

(b) State and prove any one property of torsion tensor. Also express the torsion tensor in terms of connection coeffiecients .

(c) Define a differentiable map from one manifold to another illustrated diagramatically.

(d) Define a connection preserving map. Prove that the Jacobian preserves torsion tensor and curvature tensor.

(e) If $a = A(dy \wedge dz) + B(dz \wedge dx) + C(dx \wedge dy)$, then compute the exterior derivative da.

(f) State and prove Koszul's formula. Hence derive the Christoffel symbols of first and second kind.

(g) Define Contraction map of a p - form. Prove that the contraction map of sum of two p - forms equals the sum of their contraction maps.

(h) Define the Lie - bracket of two vector fields. State and prove the Jacobi identity of Lie- bracket.

(i) Prove that if w is a one - form on a smooth manifold M, then the exterior derivative dw is $C^{\infty}(M)$-linear, that is $dw(fX, Y) = f dw(X, Y)$ for all $f \in C^{\infty}(M)$ and for all vector fields X, Y in M .

(j) From the theory of Gauss and Weingarten formulae, prove that $g(AX, Y) = g(X, AY)$.