E-29/2110 9477/NJ

BHM 502- Calculus of several variables and Improper Integrals

5th semester

Note: Attempt any four questions. All questions carry equal marks.

Q1. i) Let g be differentiable at a, with total derivative $g'(a)$. Let b=g(a) and assume that f is differentiable at b, with total derivative $f'(b)$. Then prove that the composition function $h = f \circ g$ is differentiable at a.

ii) Let $f\ and\ D_2 f$ be continuous on rectangle $[a, b] \times [c, d]$. Let p and q be differentiable on $[c, d], where\ p(y) \in [a, b]\ and\ q(y) \in [a, b]$ for each y in $[c, d]$. Let F(y)= $\int_{p(y)}^{q(y)} f(x, y)dx$, if $y \in [c, d]$. Prove that F is differentiable for all $y \in [c, d]$.

Q2. i) If both the partial derivatives $D_r f$ and $D_k f$ exist in an n-ball B(c) and if both $D_{r,k} f$ and $D_{k,r} f$ are continuous at c, then prove that $D_{k,r} f(c) = D_{r,k} f(c)$.

ii) State and prove Taylor's formula for real valued function of several variables.

Q3. i) Find the lengths of the semi-axes of the quadric surface with centre at origin has the equation

$$Ax^2 + By^2 + Cz^2 + 2Dyz + 2Ezx + 2Fxy = 1$$

ii) Show that the relation $x^2 e^y - 2y + x = 0$ defines y as a function x in the neighborhood of the point (-1,0). Find the derivative of this function for x= -1.

Q4. State and prove Inverse function theorem.

Q5. i) Show that $\int_0^\infty x^{n-1} e^{-x} dx$ is convergent if, and only if, n>0.

ii) If φ is continuous in [0,∞) and $\lim_{x \to 0} \varphi(x) = \varphi_\circ\ and\ \lim_{x \to \infty} \varphi(x) = \varphi_1$, then

$$\int_0^\infty \frac{\varphi(ax) - \varphi(bx)}{x} dx = (\varphi_\circ - \varphi_1) log \frac{b}{a}$$

Q6. i) Prove that $\int_0^\infty \frac{dx}{x^n}$ converges if and only if n>1.

ii) Discuss the convergence of $\int_0^2 \frac{dx}{x^2 - 4x + 3}$

Q7. i) Show that $\iint x^{m-1} y^{n-1} dxdy$, over the positive quadrant of the ellipse ,$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$,

is $\frac{a^m b^n}{2n} \beta\left(\frac{m}{2}, \frac{n}{2} + 1\right)$.

ii) Let f be defined and bounded on a compact interval I in R^n. Then prove that f is Riemann integrable on I if, and only if, the set of discontinuities of f in I has n-measure zero.

Q8. i) Evaluate $f(\alpha) = \int_0^\infty e^{-x^2} cos\alpha x \, dx$.

ii) If f is continuous in [a,b; c,d], then prove that $\int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$.

Q9. i) Discuss he continuity of $f(x,y)$ *at* $(0,0)$ *where* $f(x,y) = \begin{cases} \frac{2xy^2}{x^3+y^3} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

ii) Show that $f(x,y) = \sqrt{|xy|}$ is not differentiable at origin although both the first order partial derivatives exist at (0,0).

iii) If $u = x^2 + y^2 + z^2$, $v = x + y + z$, $w = xy + yz + zx$, show that the jacobian $\frac{\partial u,v,w)}{\partial(x,y,z)}$ vanishes identically.

iv) Evaluate $\iint x^2y^2dxdy$ over the domain $\{(x,y): x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}$.

v) Discuss the convergence of $\int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$.

vi) If $f: S \to R^m$ be a function defined on a set S in R^n. Let f is differentiable at c in S, then prove that f is continuous at c.