B-5/2110

ADVANCED CALCULUS -I

SEMESTER-III

TIME ALLOWED 3 Hrs

M.M 40

NOTE: The candidates are required to attempt two questions each from Section A & B Section C will be compulsory.

SECTION-A

I (a) Let f: $R^2 \to R$ be defined by f(x, y) = $\begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$

Show that f is not continuous at any point of R^2 (3)

I (b) If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, then show that $\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)^2 = \frac{9}{(x+y+z)^2}$ (3)

II State and prove Young's Theorem. (6)

III (a) Prove that $J_{f^{-1}}(\alpha, \beta) = \alpha$ for any (α, β) belonging to the range of $f(x, y)$, where

$$f(x, y) = \left(\sqrt{x^2 + y^2}, \tan^{-1}\frac{y}{x}\right).$$ (3)

III (b) Use Taylor's theorem to expand $x^2y + 3y - 2$ in powers of $x - 1$ and $y + 2$ (3)

IV Find the lengths of the axes of the section of ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ by the plane $lx + my + nz = 0$ (6)

SECTION-B

V (a) Prove that $\int_0^1 \left(\int_0^1 \frac{x-y}{(x+y)^3} dy\right) dx \neq \int_0^1 \left(\int_0^1 \frac{x-y}{(x+y)^3} dx\right) dy$ (3)

V (b) Evaluate $\int_0^\infty \int_0^x x e^{-\frac{x^2}{y}} \, dy dx$ by changing the order of integration. (3)

VI Prove that $\iint \sqrt{|y - x^2|} \, dx \, dy = \frac{3\pi+8}{6}$ over Area $A = [-1, 1] \times [0, 2]$. (6)

VII (a) Find centroid of the hemispherical region

$A = \{(x, y, z): x^2 + y^2 + z^2 \leq 1, z \geq 0\}$ where density $\mu(x, y, z) = x^2$ (3)

VII (b) Find moment of inertia of the cylinder $x^2 + y^2 \leq a^2, 0 \leq z \leq h$ with uniform mass density 1 about the z-axis. (3)

VIII (a) Find the volume of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside The cylinder $x^2 + y^2 = ax$. (3)

VIII (b) Evaluate $\iiint \frac{dxdydz}{\sqrt{x^2+y^2+(z-2)^2}}$, over region sphere $x^2 + y^2 + z^2 \leq 1$ (3)

SECTION-C

IX (a) Show that the functions $u = x + y - z, v = x - y + z, \; w = x^2 + y^2 + z^2 - 2yz$ Are not independent of one another.

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IX (b) If z= f(x, y) is a homogeneous function of x and y of degree n, then show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

IX (c) State Schwarz's Theorem.

IX (d) Shoe that point (0, 0) is a saddle point of the function $f(x, y) = x^3 + y^3 + 3xy$.

IX (e) Find the centroid of a cubic box with side 2 unit and density 1

IX (f) If a region A is defined as $A = \{(x, y): 0 \le x \le 3, 2 \le y \le 5\}$,

then show that $108 \le \iint (2x^2 + 3y^2) dx\, dy \le 837$.

IX (g) Evaluate $\iint r^2 dr d\theta$ *over the area included between circles* $r = 2\sin\theta$ *and* $r = 4\sin\theta$.

IX (h) Evaluate $\iiint (z^5 + z)\, dxdydz$ *over region* $V = \{(x, y, z): x^2 + y^2 + z^2 \le 1\}$

(2× 8 = 16)