PC-1732/M

M-32/2051

THEORY OF LINEAR OPERATORS–MM 702/AMC-411

(Semester–IV)

Time : Three Hours] [Maximum Marks : 70

Note : Attempt *two* questions each from Section A and B. Section C will be compulsory.

SECTION–A

I. All matrices representing a given linear operator $T : X \to X$ on a finite dimensional normed space X relative to various bases for X have the same eigenvalues.

II. Let X be a complex Banach space, $T \in B(X, X)$ and $p(\lambda) = \alpha_n \lambda^n + \alpha_{n-1} \lambda^{n-1} + ... + \alpha_0$. Then $\sigma(p(T)) = p(\sigma(T))$.

III. Let A be a complex Banach algebra with identity. Then the set G of all invertible elements of A is an open subset of A. Hence the subset M = A – G of all non-invertible elements of A is closed.

IV. Let X and Y be normed spaces and $T : X \to Y$ be a linear operator. Then T is compact if and only if it maps every bounded sequence $\{x_n\}$ in X onto a sequence $\{Tx_n\}$ in Y which has a convergent subsequence. (2×10=20)

SECTION–B

V. Let $T : X \to Y$ be a linear operator. If T is compact, then its adjoint operator $T^* :: Y' \to X'$ is also compact. Here X and Y are normed spaces and X' and Y' the dual spaces of X and Y.

VI. Let $T : X \to X$ be a compact linear operator on a normed space X, and let $\lambda \neq 0$. Then equations $T_\lambda x = 0$ and $T_\lambda^* f = 0$ have the same number of linearly independent solutions.

VII. Let $T : H \to H$ be a bounded self adjoint linear operator on a complex Hilbert space H. Then a number λ belongs to the resolvent set $\rho(T)$ of T if and only if there exists a number $c > 0$ such that for every $x \in H$,

$$\| T_\lambda x \| \geq c \| x \|$$

VIII. If two bounded self adjoint linear operators S and T on a Hilbert space H are positive and commute (ST = TS), then their product ST is positive. (2×10=20)

SECTION–C

IX. (a) Define Types of spectrum.

(b) Define monotone sequences and positive square root of an operator.

(c) Define spectral family.

(d) Define projection. Show that a bounded linear operator $P : H \to H$ on a Hilbert space H is a projection if P is self-adjoint and idempotent.

(e) Define Fredholm alternative.

(f) Define Biorthogonal systems.

(g) Let $T : X \to X$ be a compact linear operator and $S : X \to X$ a bounded linear operator on a normed space X. Then TS is compact.

(h) Define Resolvent set.

(i) Define Banach algebra.

(j) Define spectral radius. (10×3=30)