PC-5820/MJ

G-10/2051

ALGEBRA–I – 401
(Semester–IV)
(Syllabus May-2018)

Time : Three Hours] [Maximum Marks : 75

Note : Attempt *two* questions each from Sections A and B carrying 11¼ marks each and the entire Section C consisting of 10 short answer type questions carrying 3 marks each.

SECTION – A

I. (a) State and prove Lagrange's theorem. Write the statement of converse of Lagrange's theorem. (6)

(b) Prove that subgroup of cyclic group is cyclic. (5¼)

II. (a) Write all symmetrics of regular n-gon. (6)

(b) Prove that a group of prime power order has non-trivial centre. (5¼)

III. (a) Prove that the Kernel of a group homomorphism $\varphi : G \to G'$ is a normal subgroup of G. Moreover, φ is injective if and only if Ker $\varphi = \{1\}$. (6)

(b) Prove that if the commuting elements a, b of group G have coprime orders k and l, then ab has order kl. (5¼)

IV. (a) State and prove first isomorphism theorem of group. (5¼)

(b) Prove that $\mathbb{Z}_{18}$, the group of integers modulo 18, is cyclic. Find all generators of $\mathbb{Z}_{18}$. (6)

SECTION – B

V. State and prove Sylow's first theorem. (11¼)

VI. (a) Prove that a finite group is a p-group if and only if its order is power of p. (6)

(b) Write the class equation of a group of order 25. (5¼)

VII. (a) State and prove Cayley's theorem. (5¼)

(b) Show that A_4, the alternating group of degree 4, has no element of order six. (6)

VIII. Prove that every finite rotation group in $\mathbb{R}^3$ is either cyclic or dihedral or tetrahedral or octahedral or icosahedral. (11¼)

SECTION – C

IX. (a) Prove that a subgroup of index 2 is normal in it.

(b) Find all generators of cyclic group $\mathbb{Z}$ and $\mathbb{Z}_8$.

(c) For any two elements a, b of finite group G, prove that $0(ab) = 0(ba)$.

(d) Prove that every subgroup of an Abelian group is normal in it. Is the converse true? Justify.

(e) Find the index of $n\mathbb{Z}$ in $\mathbb{Z}$.

(f) Find all Sylow 2-subgroups of S_3. Verify that there are conjugate to one another.

(g) Prove that a Sylow p-subgroup of finite group G is normal, if and only if p is unique Sylow p-subgroup.

(h) Show by an example that if H is a normal subgroup of G and K is a normal subgroup of H, then K may not be a normal subgroup of G.

(i) Prove that A_n is a normal subgroup of S_n of index 2, $n \geq 2$.

(j) Prove that the homomorphic image of cyclic group is cyclic. (3×10=30)