Total Pages : 4

PC-746/MH

CS/2051

ALGEBRA–II

Paper–III

(Semester–VI)

Time : Three Hours] [Maximum Marks : 40

Note : Attempt *two* questions each from Section A and B. Section C will be compulsory.

SECTION–A

I. Prove that there exist a basis for each finite dimensional vector space. (6)

II. Let V be the set of all $n \times n$ skew-symmetric matrices over field **R**. Let vector addition and scalar multiplication be defined as usual addition of matrices and multiplication of a scalar with a matrix. Show that V is a vector space over **R.** (6)

III. (a) Find the basis and dimension of solution space of given system of linear equations :

$$x + 2y - 2z + 2s - t = 0$$

$$x + 2y - z + 3s - 2t = 0$$

$$2x + 4y - 7z + s + t = 0.$$ (3)

(b) If X_1, X_2, X_3 ...., X_r are linearly independent column vectors of order $m \times 1$ over a field F and A is $m \times m$ singular matrix over F, then show that AX_1, AX_2, AX_3, AX_r are linearly dependent over F. (3)

IV. (a) Let W be a subspace of the vector space V over a field F. If set

$$\{v_1 + W, v_2 + W, v_3 + W, \ldots\ldots\ldots, v_n + W)$$

is a Linear Independent subset of V/W, show that Set $\{v_1, v_2, v_3 \ldots.., v_n\}$ linear Independent subset of V. (3)

(b) Let V(R) be a vector space of all $n \times n$ matrices over a field F. Let W and U be subspaces of V(R) of symmetric and skew-symmetric matrices respectively. Show that $V = W \oplus U$. (3)

SECTION–B

V. State and prove Sylvester's law of nullity. (6)

VI. Let T be a linear operator defined on R^3 defined by

$$T(x, y, z) = (x + y, y + z, z).$$

Find the characteristic and minimal polynomial of T. (6)

VII. (a) Let V be a finite dimensional vector space over a field F and T; $V \to V$ be a linear operator. Show that Range (T) $\cap$ Ker (T) = {0} if and only if for all $v \in V$, $T(T(v)) = 0 \Rightarrow T(v) = 0$. (3)

(b) Let T be a linear operator defined on R^2 defined by

$T(x, y) = (x + 2y, 3x + 4y)$.

Find $p(T)$, where $p(T) = t^2 - 5t - 2$. (3)

VIII. If the matrix of a linear operator T on R^3 relative to the standard basis is

$$\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix},$$

then find the matrix T relative to the basis

$B = \{(1, 2, 2), (1, 1, 2), (1, 2, 1)\}$

Hence verify that $[T; B]\,[v; B] = [T(v); B]\ \forall\ v$ E $\in R^3$. (6)

SECTION–C

IX. (a) If W is a subspace of a finite dimensional vector space V over a field F, then prove that dim W ≤ dim V.

(b) Prove that intersection of two subspaces of a vector space is also a subspace.

(c) Let V be a vector space over a field F. Suppose a finite subset $S = \{x_1, x_2, x_3, ..., x_n)$ of non-zero elements of V is linearly dependent then prove that some element say x_k of S can be written as a linear combination of remaining elements of S.

(d) Find the dual basis of the usual basis of R^3.

(e) Give an example of a linear operator T such that $T \neq 0, T^2 \neq 0, T^3 \neq 0, ..., T^{n-1} \neq 0$, but $T^n = 0$.

(f) Find a Linear Transformation $T : R^3 \rightarrow R^3$ whose range space is spanned by (1, 0, –1) and (1, 2, 2).

(g) If $B_1 = \{(1, 0), (0, 1)\}$ and $B_2 = \{(1, 2), (-3, -5)\}$ are bases of R^2, find transition matrix from B_2 to B_1.

(h) Show that there is no non-singular linear transformation $T : R^4 \rightarrow R^3$. (2×8=16)